

Operators and vacua of $\mathcal{N} = 1$ field theories

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LPTENS-09/41

Summary. — We review the idea of Hilbert Series as a tool to study the moduli space and the BPS operators of four dimensional $\mathcal{N} = 1$ supersymmetric field theories. We concentrate on the particular case of $\mathcal{N} = 1$ superconformal field theories living on N D3 branes at toric Calabi-Yau singularities. The main claim is: it is possible to write down explicit partition functions counting all the local BPS operators for generic N number of branes, and obtain important informations about the BPS operators, the moduli space and the dual geometry.

PACS: 11.25.Tq, 11.25.Uv, 11.25.Hf

1. – Introduction

In this paper I would like to give an overview of the study of supersymmetric vacua and BPS operators of $\mathcal{N} = 1$ supersymmetric field theories using the idea of Hilbert Series (HS). I will consider as explicit example the super-conformal field theories (SCFT) living on a stack of N D3 branes at the tip of the Calabi-Yau (CY) six dimensional conical singularity $\mathcal{X} = C(H)$.

This kind of “new” approach to $\mathcal{N} = 1$ supersymmetric field theories was born from a set of very simple observations: every $\mathcal{N} = 1$ supersymmetric field theory has a special subset of operators: the local BPS operators, we would like to have a generating function (HS) counting them according to their global charges; the local BPS operators are the holomorphic functions over the moduli space: once we know the BPS spectrum of the theory we also know its moduli space.

To write down explicit HS counting all the local BPS operators of a non-abelian field theory is a rather complicated task and we will refer the interested reader to the original literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In this review paper we would just like to point out that the HS contains a lot of informations related to the field theory, and it is worthwhile to study them.

In particular the HS contains the structure of generators and relations of the chiral ring and it knows about the dimension, the symmetries and the algebraic geometric structure of the complete moduli space.

In the special case of SCFT living on D-branes at singularities we can calculate the HS at weak coupling, using field theory, or at strong coupling using string theory and compare the two results. Moreover the HS knows about the central charge and the exact R-charges of the elementary fields, it contains part of the KK spectrum in supergravity and the values of the volume of the supergravity compactification manifold H and of some of its supersymmetric three cycles.

With this short introduction we hope to have motivated the reader to know more about HS. In the following we will at first introduce our favorite example of supersymmetric theories: the SCFT living on N D3 branes at the tip of the CY singularity $\mathcal{X} = C(H)$; we will then divide the study of their chiral ring in $N = 1$ and $N > 1$: we will introduce the idea of Master Space (MS) and we will illustrate the generic structure of the HS for $N=1$; we will comment on some of the properties of the MS and will summarize some of the informations we could obtain from the HS. We will then study the $N > 1$ case: we will introduce the idea of Plethystic Exponential (PE) and we will write down the generic form of the HS counting all the local BPS operators for generic N for every SCFT on D3 branes at toric CY singularity; we will comment on some of its properties and we will explain some of the informations we can obtain from the HS regarding the complete moduli space of the theory for generic N . We will finally analyze, with an explicit example, some of the generalities explained all along the paper, and we will conclude with some comments about other possible applications.

This paper is based on the author's PhD thesis [11], presented at the INFN conference, 21/09/2009, Rome, for the "Premio Fubini 2009". See also [12].

2. – D3 branes at singularities

In this paper we will concentrate on the SCFT living on N D3 branes at the tip of CY singularity $\mathcal{X} = C(H)$. The main reasons for this choice are: they are simple enough to obtain explicit and general results, and they are very interesting for at least two reasons: they are the basis of very interesting phenomenological models that embed generalizations of the MSSM and SUSY breaking mechanisms in string theory; and they all have a strong coupling description in the AdS/CFT correspondence: type IIB string theory on $AdS_5 \times H$.

The low energy dynamics of D3 branes at \mathcal{X} is a quiver gauge theory [13]: it is an $\mathcal{N} = 1$ supersymmetric field theory with a manifold of IR superconformal fixed points, it has gauge group $\prod_i^g SU(N_i)$, chiral bifundamental matter fields, and a particular form for the superpotential. For simplicity we will always assume that the singularity \mathcal{X} is toric, meaning that it admits at least a $U(1)^3$ isometry. In this case there is a rather complete dictionary between the field theory and the dual geometry.

The natural presence of chiral fields is what makes these theories phenomenologically interesting. Indeed generalizing a bit this setup one can reproduce very interesting MSSM-like scenario, SUSY-breaking, renormalization group flows and confinement, moreover they are nice models to study non-perturbative instanton processes that could produce some perturbatively forbidden couplings like some mass terms or Yukawa interactions.

On the other end the AdS/CFT correspondence claims that the field theory living on

N D3 branes at the $\mathcal{X} = C(H)$ singularity is dual, in a strong/weak coupling sense, to type IIB string theory propagating on the $AdS_5 \times H$ background with N unit of flux: $\int_H F_5 = N$.

We would like to study two main objects of these field theories: their moduli spaces and their BPS operators for generic N number of D branes. The moduli space \mathcal{M}_N is the variety of expectation values of the scalar fields that do not break the supersymmetry; the local BPS operators are local gauge invariant operators \mathcal{O} killed by half of the supercharges $\bar{D}\mathcal{O} = 0$.

The gauge group factors are $SU(N)$ groups and they admit two kinds of invariant tensors: δ_j^i and $\epsilon_{i_1, \dots, i_N}$. The gauge invariant operators constructed contracting the elementary bifundamental fields with the δ are trace-like and are called mesonic operators, while the operators obtained using the ϵ are determinant-like and are called baryonic operators. These are the two kinds of BPS operators we want to study.

The global symmetries of these SCFTs come from the geometry of \mathcal{X} . There exist two different types of symmetries: the mesonic ones coming from the isometries of \mathcal{X} and the baryonic ones under which only the determinant like operators are charged and coming from the topology of three cycles of H.

The setup of D branes at singularity has a set of nice properties that will make easier our analysis. A first crucial observation is that at least a part of the moduli space of the gauge theory is explicitly realized in the geometry. Indeed some of the scalar degrees of freedom parameterizing the moduli space \mathcal{M} must also parameterize the space transverse to the branes. In the abelian case we just have one D3 brane and it is just a free point in the transverse geometry, hence the moduli space of the theory must contain at least the six dimensional singularity \mathcal{X} . In the generic non-abelian case we have N D3 branes that, being mutually supersymmetric, will behave as a set of N non-interacting particles on \mathcal{X} . Because the ground state of the D3 branes is bosonic we have a system of N bosons and the non-abelian moduli space must contain the N-times symmetric product of the transverse singularity: $\text{Sym}^N(\mathcal{X}) = \mathcal{X}^N / \Sigma_N$, where Σ_N is the group of permutations of N objects acting on the N factors \mathcal{X} .

These nice properties allow for geometrical intuition and subdivide the problem in $N=1$ and $N > 1$.

3. – Hilbert Series and Master Space: the $N=1$ case

Our path to understand the complete spectrum of local BPS operators and the full moduli space \mathcal{M}_N will be somehow an unusual path: we want to compute HS!

The set of holomorphic functions over the moduli space \mathcal{M} is exactly the set of all local BPS operators of the gauge theory. We want to construct a function that encode the informations of the chiral ring of the theory. We proceed in the following way. The gauge theory has generically $g + 2$ global abelian symmetries: $U(1)^2 \times U(1)_R$ coming from the isometries of \mathcal{X} , the mesonic symmetries, and $U(1)^{g-1}$ baryonic symmetries: anomalous and non-anomalous, coming from the $U(1)$ factors inside the $U(N_i)$ of the UV description of the quiver gauge theory that decouple in the IR superconformal fixed points; the ones associated to the three cycles in H are the non-anomalous ones.

We want to count the BPS operators according to their charges under these various $U(1)$ s. Let us introduce a set of $g+2$ chemical potentials t_i with $i = 1, \dots, g+2$. We would

like to have a function that once expanded for small values of the chemical potentials

$$(1) \quad H(t; \mathcal{M}) = \sum_{i_1, \dots, i_{g+2}} c_{i_1, \dots, i_{g+2}} t_1^{i_1} \dots t_{g+2}^{i_{g+2}}$$

gives the numbers $c_{i_1, \dots, i_{g+2}}$ of BPS operators with charges i_j under the corresponding global abelian symmetries. In the mathematical literature this function is called the Hilbert Series of \mathcal{M} , and it is actually counting the holomorphic function over the algebraic variety \mathcal{M} according to their charges under the action of its symmetry group.

In general \mathcal{M} has a nice symplectic quotient description: it is the quotient of the algebraic variety \mathcal{F}^\flat defined by the derivatives of the superpotential with respect to the elementary fields (F-flatness conditions) by the complexified gauge group G :

$$(2) \quad \mathcal{M} \simeq \mathcal{F}^\flat / G .$$

This description of the moduli space clearly individuate a parent space \mathcal{F}^\flat , that we will call the Master Space (MS) of the theory [9], and it will be very useful to study the full moduli space \mathcal{M}_N for generic N . The MS is a gauge dependent space that can be associated to every $\mathcal{N} = 1$ field theory, it is simpler than the full moduli space and we could start enjoying to classify its properties. In the case of D3 branes at \mathcal{X} it turns out that the knowledge of just a subset of the MS for one brane is enough to obtain the HS counting BPS operators for generic N !

Let us start discussing the abelian $N=1$ case. The plan will be the following: we will fully characterize the moduli space in the abelian case and we will then explain the generic structure of the HS (1) for one brane. In the next section we will discuss the non-abelian case.

With just one brane we are left with a set of $SU(1)$ groups, the gauge dynamics is trivial and $\mathcal{M}_1 = \mathcal{F}_1^\flat$. The MS \mathcal{F}_1^\flat is the complete moduli space of the theory and contains as a sub set (the locus where all the baryonic operators have zero vacuum expectation value) the geometrical six dimensional transverse singularity \mathcal{X} .

The master space is a $(\mathbb{C}^*)^{g-1}$ fibration over \mathcal{X} with the fiber structure induced by the baryonic symmetries. It is nice to observe that \mathcal{F}_1^\flat has a very rich structure:

- $\mathcal{M}_1 = \mathcal{F}_1^\flat$ is a $g+2$ dimensional toric variety, and can be completely characterized in term of algebraic geometry. In particular \mathcal{F}_1^\flat is generically reducible in
 - a $g+2$ dimensional component ${}^{\text{Irr}}\mathcal{F}_1^\flat$ that is a toric Calabi Yau cone,
 - and a set of lower dimensional generically linear components L_i .

It is possible to show that ${}^{\text{Irr}}\mathcal{F}_1^\flat$ has the rigid structure already explained and moreover it has nice transformation properties under Seiberg dualities and a defined meaning in the dual string theory/strongly coupled description. A case by case analysis shows instead that the lower dimensional spaces L_i are less constrained, even if one can show that they could parameterize a subclass of RG flows of the theory. For these reasons we would like to concentrate on ${}^{\text{Irr}}\mathcal{F}_1^\flat$, that indeed turn out to be the variety containing the necessary informations to obtain the BPS spectrum for generic N .

We know that ${}^{\text{Irr}}\mathcal{F}_1^\flat$ is the fibration: $(\mathbb{C}^*)^{g-1} \rightarrow \mathcal{X}$, where the fiber are generated by the baryonic symmetries of the theory, while the isometries of the base space \mathcal{X} are the mesonic symmetries. Let us define \vec{B} the vector of non-anomalous baryonic charges

and q the generic chemical potential associated both to the mesonic and the baryonic symmetries. It is natural to guess that the HS for ${}^{\text{Irr}}\mathcal{F}_1^b$ admits a decomposition in sectors having a different set of baryonic charges:

$$(3) \quad H(q; {}^{\text{Irr}}\mathcal{F}_1^b) = \sum_{\vec{B}} m(\vec{B}) g_{1,\vec{B}}(q; \mathcal{X})$$

where for every fixed values of \vec{B} the function $g_{1,\vec{B}}(q; \mathcal{X})$ counts all the operators with fixed non-anomalous baryonic charges and all the possible mesonic charges; and $m(\vec{B})$ are some multiplicities: the number of different operators with the same values all non-anomalous charges. If we consider only the non-anomalous baryonic charges the multiplicity $m(\vec{B})$ is in general a complicated function, if instead we consider also the anomalous charges the multiplicity is always one. The non-anomalous charges are of course the good quantum variables, but we will see that the addition of the anomalous charges will be useful to study interesting properties of the non-abelian spectrum and of the non-abelian moduli space.

One can verify that the HS actually decomposes as in (3). We can compute it in at least two ways. We can calculate the left end side of (3) using the field theory and some algebraic geometric techniques, and then perform some kind of generalized Laurent expansion in the baryonic charges [6, 7]. Alternatively we can compute it directly on the gravity side [2, 3, 4, 7]. To every BPS operators one can associate a quantum state of a D3 brane wrapped over a three cycle in H. If the three cycle is trivial the corresponding operator will be a mesonic operator; if the three cycle is non-trivial the associated operator is a baryonic operator. There exist an infinite number of states of D3 branes wrapped in H. Indeed every section of every possible line bundle over \mathcal{X} individuates a supersymmetric embedding of a D3 brane. The topology of the cycle fixes the baryonic charges \vec{B} and using an index theorem we can compute $g_{1,\vec{B}}(q; \mathcal{X})$ counting all the sections of the associated line bundle according to their charges under the isometry of \mathcal{X} . The multiplicities $m(\vec{B})$ can be computed instead using the Kahler cone and the quiver combinatoric. We address the interested reader to the original papers, here we just want to point out that (3) can be computed for every SCFT on D3 branes at singularities both in field theory and in the dual gravity side, and the two results agree.

$H(q; {}^{\text{Irr}}\mathcal{F}_1^b)$ contain very interesting informations:

- $g_{1,0}(t; \mathcal{X})$ is the mesonic partition function, it contains the structure of the algebraic equations defining the variety \mathcal{X} and part of the KK spectrum of the supergravity reduction over H;
- taking a limit and minimizing $g_{1,\vec{B}}(t; \mathcal{X})$, for generic \vec{B} , we can obtain the volume of H, and for $\vec{B} \neq 0$ the volumes of all the possible non-trivial three cycles C_3 in H. Moreover using the AdS/CFT correspondence we obtain the value of the central charge a and of the R-charge for every elementary field at the strong coupling fixed point.

4. – $N > 1$: Hilbert Series, Master Space and Plethystic Exponential

We would like now to discuss the BPS operators in the non-abelian case. The step from $N=1$ to generic N is rather non-trivial due to the non-abelian dynamics and the non-trivial finite- N relations among the BPS operators. Moreover a direct computation

of the HS in the field theory side becomes computationally prohibitive around $N=3$. In this section we want to point out that the knowledge of the geometrical structure of ${}^{\text{Irr}}\mathcal{F}_1^\flat$, namely its symmetries and its HS decomposition (3), is enough to obtain the HS counting all the local BPS operators for generic N and to obtain some interesting informations about the non-abelian moduli space \mathcal{M}_N [7, 9].

As we have already explained the geometric intuition for the D3 brane system induce to think that to pass from $N=1$ to generic N we need to consider some sort of N -times symmetric product. The direct analysis of the generic BPS operator and the geometric quantization of the dual D3 brane state in presence of N units of flux, support this guess and define the precise meaning of the N -times symmetric product.

The moduli space \mathcal{M}_N is non-toric and non-CY. It can be described as a \mathbb{C}^* fibration over the mesonic moduli space: $\mathcal{M}_N \simeq (\mathbb{C}^*)^{g-1} \rightarrow \text{Sym}^N \mathcal{X}$. To explicitly write down the generic form of the HS of \mathcal{M}_N , it is useful to introduce a special function called the Plethystic Exponential: $\text{PE}[\dots]$. It is the generating function for the order N -symmetric products. Indeed the Plethystic Exponential (PE) evaluated on a function g_1 , counting a certain set of operators, generates the functions g_N counting all the possible N -times symmetric product of the operators counted by g_1 .

Let us introduce the chemical potential ν , we define the plethystic exponential PE_ν as:

$$(4) \quad \text{PE}_\nu[g_1(t)] = \exp \left(\sum_{r=1}^{\infty} \frac{\nu^r g_1(t^r)}{r} \right) = \sum_{N=0}^{\infty} g_N(t) \nu^N.$$

Using the PE we will easily pass from $N=1$ D brane to generic N D branes. Moreover the PE has an inverse function called the Plethystic Logarithm ($\text{PL}=\text{PE}^{-1}$): acting with the PE^{-1} on a generating function we obtain the generating series for the generators and the relations in the chiral ring.

To pass to the generic N case we just need to construct the partition function counting all the possible N times symmetric products of the functions counted by $g_{1,\vec{B}}(t; \mathcal{X})$. The PE plays exactly this role.

The Hilbert series counting the BPS operators for arbitrary number N of branes is obtained from (3) applying the PE to every fixed \vec{B} sector:

$$(5) \quad \sum_{\nu=0}^{\infty} \nu^N H(q; \mathcal{M}_N) = \sum_{\vec{B}} m(\vec{B}) \text{PE}_\nu[g_{1,\vec{B}}(t; \mathcal{X})]$$

where $H(q; \mathcal{M}_N)$ is the Hilbert series counting the BPS operators for N D3 branes. Once we have $H(q; \mathcal{M}_N)$ we can expand it in power of the chemical potential q and obtain the numbers of independent local BPS operators for every values of the charges. We could also look for large charges limits to study thermodynamical properties of the quiver theory [5], or we can apply the PL to $H(q; \mathcal{M}_N)$ and study the structure of generators and relations in the chiral ring [6].

Summarizing: the study of the MS for just one brane gave us the tools to compute partition functions counting BPS operators for arbitrary number N of branes for every SCFT living on N D3 branes at the tip of \mathcal{X} .

The nice surprises are not finished yet. The BPS operators are the holomorphic functions over the moduli space \mathcal{M}_N and knowing the complete spectrum of BPS operators is equivalent to know the full moduli space. Following this line of thinking it is clear that

$H(q; \mathcal{M}_N)$ can teach us something about \mathcal{M}_N . Once again the MS can help to obtain informations about \mathcal{M}_N .

Sometimes the MS $\text{Irr}\mathcal{F}_1^\flat$ for just one D3 brane has some symmetries not manifest in the UV Lagrangian describing the field theory; we call them hidden symmetries. These symmetries are typically non-abelian extensions of some anomalous or non-anomalous baryonic $U(1)$ symmetries.

It is nice to observe that the symmetries of the moduli space \mathcal{M}_N can be naturally studied with the help of the HS (5). Indeed sometime it is possible to reorganize the spectrum of BPS operators in multiplets of the hidden symmetries in such a way that the HS for the $N = 1$ master space will be explicitly written in term of characters $\chi_{\vec{l}}(p)$ of these symmetries. If it is the case, it would imply that the BPS operators are in representation of the symmetries of $\text{Irr}\mathcal{F}_1^\flat$. We just learned how to pass from one brane to generic N . If we are able to reorganize the generating functions $H(q; \mathcal{M}_N)$, for the non-abelian case, in representations of the global symmetries guessed from the abelian case, we could claim that both the BPS operators and the complete moduli space \mathcal{M}_N , for generic N , have the hidden symmetries of the $N=1$ Master Space $\text{Irr}\mathcal{F}_1^\flat$.

Indeed in a case by case analysis it is possible to show that, if we introduce a set of chemical potentials p_i , $i = 1, \dots, g + 2$ parameterizing the Cartan sub-algebra of the hidden symmetries group, it is sometime possible to rewrite the HS of \mathcal{M}_N in such a way that it will have a nice expansion in characters:

$$(6) \quad H(p; \mathcal{M}_N) = \sum_{\vec{l}} \chi_{\vec{l}}(p)$$

We could try to use the tools we just developed to study other problems, and we will comment about some applications in the conclusions. Right now we would like to give an easy explicit example to illustrate some of the general points explained in the main text.

5. – An Example: the Cone over the Zeroth Hirzebruch Surface

Till now the discussion was quite abstract, let us try to give an explicit example. We choose to study the low energy field theory living on N D3 branes at the tip of the complex cone over $\mathbb{P}^1 \times \mathbb{P}^1$. This particular surface is called the zeroth Hirzebruch surface \mathbb{F}_0 . The field theory can be nicely represented by the quiver diagram in Figure 1. To every small circle we associate an $SU(N)$ gauge group factor, and to every arrow from $SU(N)_i$ to $SU(N)_j$ we associate a chiral field X_{ij} charged under the fundamental of $SU(N)_i$ and the anti-fundamental of $SU(N)_j$. The superpotential of the theory is: $W = \epsilon_{ij} \epsilon_{pq} A_i B_p C_j D_q$. Let us start with the abelian $N=1$ case. The MS \mathcal{F}_1^\flat can be readily found taking the derivatives of the superpotential. There are three irreducible pieces:

$$(7) \quad \mathcal{F}_1^\flat = \text{Irr}\mathcal{F}_1^\flat \cup L^1 \cup L^2,$$

with

$$(8) \quad \begin{aligned} \text{Irr}\mathcal{F}_1^\flat &= \mathbb{V}(B_2 D_1 - B_1 D_2, A_2 C_1 - A_1 C_2) \\ L^1 &= \mathbb{V}(C_2, C_1, A_2, A_1) \\ L^2 &= \mathbb{V}(D_2, D_1, B_2, B_1) . \end{aligned}$$

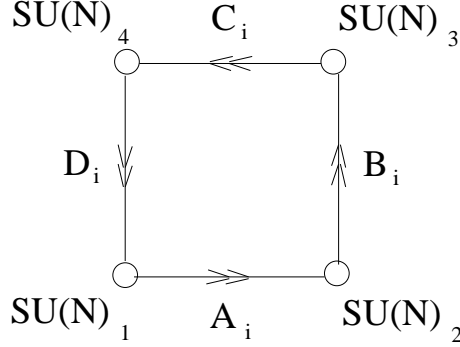


Fig. 1. – The quiver for the \mathbb{F}_0 field theory.

where $\mathbb{V}(\dots)$ means the locus of zero of the polynomials inside the brackets. The MS has a rich structure. It is reducible in three irreducible components. The $\text{Irr}\mathcal{F}_1^b$ is $g + 2 = 4 + 2 = 6$ dimensional and it is a cone. The complete intersection $xy = wz$ is a well known toric CY three-fold, it is called the conifold and it has $SU(2) \times SU(2) \times U(1)$ isometry group. $\text{Irr}\mathcal{F}_1^b$ is the product of two conifold and it is indeed a six dimensional CY toric cone. If we give weight t to all the eight basic fields in the quiver the HS of $\text{Irr}\mathcal{F}_1^b$ is:

$$(9) \quad H(t; \text{Irr}\mathcal{F}_1^b) = \frac{(1+t)^2}{(1-t)^6}.$$

Let us analyze the symmetries of the field theory. From the form of the quiver and of the superpotential we can see that there are two mesonic non-abelian $SU(2)$: the fields A and C are doublets of the first $SU(2)$, while B and D are doublets of the second $SU(2)$. On top of these there are also a non-anomalous $U(1)_R$ R-symmetry under which all the elementary fields have the same $1/2$ charge, and a non-anomalous $U(1)_B$ baryonic symmetry with charge $+1$ for the fields A and C , and -1 for the fields B and D . At the classical level there are other two $U(1)$ s that are actually anomalous in the quantum theory. We conclude that the classical symmetry of the field theory is: $SU(2)^2 \times U(1)^4$. The $\text{Irr}\mathcal{F}_1^b$ is the product of two conifold and has $SU(2)^4 \times U(1)^2$ symmetry. Namely the symmetry of the MS is bigger than the symmetry of the Lagrangian describing the SCFT. We call the new symmetries the hidden symmetries of the theory.

We would like to decompose the HS of $\text{Irr}\mathcal{F}_1^b$ in sectors with different baryonic charges as in (3). We introduce the chemical potentials t and b for the R and non-anomalous baryonic charges, and we define the characters $[n] \times [m] \times [p] \times [q] = [n, m, p, q]$ for the $SU(2)^4$ representations: the first two brackets for the $SU(2)^2$ mesonic symmetries, and the second two brackets for the $SU(2)^2$ hidden symmetries. We can organize the four fields A, C in the $[1, 0, 1, 0]$ representation and the four fields B, D in the $[0, 1, 0, 1]$ representation. The HS (9) can be refined with all the chemical potentials and can be written explicitly in $SU(2)^4 \times U(1)^2$ covariant form:

$$(10) \quad H(q; \text{Irr}\mathcal{F}_1^b) = \sum_{\beta, \beta'=0}^{\infty} [\beta, \beta', \beta, \beta'] t^{\beta+\beta'} b^{\beta-\beta'}$$

To write down the decomposition (3) we just need to explicitly write the $SU(2)^4$ characters and set to one the chemical potentials associated to the $SU(2)^2$ hidden symmetry group. This last operation would generate the multiplicities $m(\vec{B})$.

Using the completely refined decomposition of the HS for $\text{Irr}\mathcal{F}_1^b$ we can obtain the expression for the HS counting all the local BPS operators for generic N , according to their symmetries. The final result is:

$$(11) \quad \sum_{\nu=0}^{\infty} \nu^N H(q; \mathcal{M}_N) = \sum_{\beta, \beta'=0}^{\infty} [0, 0, \beta, \beta'] \left(PE_{\nu} \left[\sum_{n=0}^{\infty} [2n + \beta, 2n + \beta', 0, 0] t^{4n+\beta+\beta'} b^{\beta-\beta'} \right] - PE_{\nu} \left[\sum_{n=1}^{\infty} [2n + \beta, 2n + \beta', 0, 0] t^{4n+\beta+\beta'} b^{\beta-\beta'} \right] \right).$$

Note that the first PE contains all the terms in the second PE and hence all the coefficients in the expansion are positive. This is the explicit demonstration that for generic N the chiral spectrum organizes into representations of $SU(2)^4 \times U(1)^2$ and, as a consequence, also the moduli space of the non-abelian theory with generic rank N has symmetry $SU(2)^4 \times U(1)^2$. Once again if we want to count the operators according to the non-anomalous symmetries as in (5) it is enough to explicitly write down the characters and then set to one the chemical potentials associated to the hidden symmetries.

6. – Conclusions

We tried an alternative algebraic geometrical approach to four dimensional $\mathcal{N} = 1$ supersymmetric field theories. We make strong use of the concept of Master Space, Hilbert Series and Plethystic Exponential. For the specific setup of D3 branes at CY singularities we got as a result the control over the spectrum of BPS operators and its symmetry properties for generic N ; the complete description for the moduli space for one brane, and interesting properties for generic N .

There are many other applications or generalizations of the techniques and results we discussed in these notes. For example it is possible to extend the counting procedure to include also the fermionic operators [1, 8], to study marginal deformations of the field theories [14] or to study the behavior of HS and MS under quantum dualities like Seiberg duality [10].

It is important to observe that, even if the tools we developed work very well for the gauge theories living on D3 branes at CY three-fold, they can easily be applied to other interesting field theories. Some examples are the study of SQCD-like theories [15] or, more recently, the study of the three dimensional conformal field theories living on M2 branes at CY four-fold singularities [16, 17].

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I would like to thanks my collaborators for the nice feeling all along these years and the INFN and SIF that gave me the opportunity to write this short review. D. F. is supported by CNRS and ENS Paris.

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